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2009 J. Phys. A: Math. Theor. 42 095212

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Randers metrics of Berwald type on four-dimensional hypercomplex Lie groups

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Received 15 September 2008, in final form 2 January 2009

Published 5 February 2009

Online at stacks.iop.org/JPhysA/42/095212

Abstract

In the present paper we study Randers metrics of Berwald type on simply connected four-dimensional real Lie groups admitting invariant hypercomplex structure. On these spaces, the Randers metrics arising from invariant hyper-Hermitian metrics are considered. Then we give explicit formulae for computing the flag curvature of these metrics. By this study we construct two four-dimensional Berwald spaces, one of them with a non-negative flag curvature and the other one with a non-positive flag curvature.

PACS numbers: 02.40.Ma, 0.20.Vh

Mathematics Subject Classification: 53C15, 58B20, 53B35, 53C60

1. Introduction

Hyper-Kähler geometry with torsion (HKT-geometry) is an important field in differential geometry which has many applications in theoretical physics. These structures appear in the supersymmetric sigma model and the study of black holes (see [17, 22]). The background object of HKT-geometry is hyper-Hermitian manifolds. Also one of the other applications of hypercomplex structures is in topological quantum field theory (see [20]).

A special class of hyper-Hermitian manifolds are Lie groups. On Lie groups, we can consider those hypercomplex structures and the hyper-Hermitian (Riemannian) metrics which are invariant under the left action of G on itself. These spaces have been studied by many mathematicians and many interesting results have been obtained by them. Barberis has given a full classification of flat homogeneous hypercomplex manifolds in dimension four (see [6]). In [7], Barberis and Dotti have shown that the tangent bundle TG of a Lie group G carrying a flat torsion-free connection ∇ and a parallel complex structure possesses a hypercomplex structure. They have also studied the construction of hypercomplex structures on a class of three step solvable groups in [8]. Invariant Abelian hypercomplex structures on eight-dimensional nilpotent Lie groups have been studied in [12], in which Dotti and Fino have

proved that a group N admitting such a structure is either Abelian or an Abelian extension of a group of type H . Also they have given a description of all eight-dimensional simply connected nilpotent Lie groups carrying a left invariant hypercomplex structure in [13]. Fino has studied the existence of conformally hyper-Kähler metrics and hyper-Kähler with torsion (HKT) structures on the cotangent bundle of hypercomplex four-dimensional Lie groups in [16]. In [21], Pedersen and Swann have shown that locally hypercomplex four-manifolds are precisely the Einstein–Weyl structures with self-dual Weyl curvature, self-dual skew-Ricci tensor and zero symmetric Ricci tensor. In [18], Gribachev and Manev have introduced isotropic hyper-Kähler manifolds and have constructed a 4-parametric family of four-dimensional manifolds of this type on a Lie group. Four-dimensional simply connected real Lie groups admitting invariant hypercomplex structures equipped with left invariant hyper-Hermitian (Riemannian) metrics are classified by Barberis (see [4, 5]). Also we completely described the Levi-Civita connection and sectional curvature of these metrics in [27].

On the other hand, Finsler metrics which are a generalization of Riemannian metrics have found many applications in physics [1, 2]. Invariant Finsler metrics on homogeneous spaces and Lie groups are interesting Finsler metrics which have been studied in recent years [10, 11, 14, 15, 24–26].

In this paper we study left invariant Randers metrics (a special Finsler metric) which arise from left invariant hyper-Hermitian (Riemannian) metrics and parallel left invariant vector fields on simply connected four-dimensional real Lie groups admitting invariant hypercomplex structure. We give the explicit formula for computing flag curvature, a generalization of sectional curvature for Finsler metrics, of such Randers metrics. In this way we introduce two special complete Randers spaces of Berwald type, one of them has a non-negative flag curvature and the other one has a non-positive flag curvature.

2. Preliminaries

Suppose that M is a $4n$ -dimensional manifold. Also let $J_i, i = 1, 2, 3$, be three fiberwise endomorphism of TM such that

$$J_1 J_2 = -J_2 J_1 = J_3, \tag{2.1}$$

$$J_i^2 = -Id_{TM}, \quad i = 1, 2, 3, \tag{2.2}$$

$$N_i = 0, \quad i = 1, 2, 3, \tag{2.3}$$

where N_i is the Nijenhuis tensor (torsion) corresponding to J_i defined as follows:

$$N_i(X, Y) = [J_i X, J_i Y] - [X, Y] - J_i([X, J_i Y] + [J_i X, Y]), \tag{2.4}$$

for all vector fields X, Y on M . Then the family $\mathcal{H} = \{J_i\}_{i=1,2,3}$ is called a hypercomplex structure on M .

In fact three complex structures J_1, J_2 and J_3 on a $4n$ -dimensional manifold M form a hypercomplex structure if they satisfy relation (2.1) (since an almost complex structure is a complex structure if and only if it has no torsion, see [19], p 124).

A Riemannian metric g on a hypercomplex manifold (M, \mathcal{H}) is called hyper-Hermitian if $g(J_i X, J_i Y) = g(X, Y)$, for all vector fields X, Y on M and $i = 1, 2, 3$.

A hypercomplex structure $\mathcal{H} = \{J_i\}_{i=1,2,3}$ on a Lie group G is said to be left invariant if for any $a \in G$,

$$J_i = Tl_a \circ J_i \circ Tl_{a^{-1}}, \tag{2.5}$$

where Tl_a is the differential function of the left translation l_a .

Let M be a smooth n -dimensional manifold and TM be its tangent bundle. A Finsler metric on M is a non-negative function $F : TM \rightarrow \mathbb{R}$ which has the following properties:

- (i) F is smooth on the slit tangent bundle $TM^0 := TM \setminus \{0\}$,
- (ii) $F(x, \lambda y) = \lambda F(x, y)$ for any $x \in M, y \in T_x M$ and $\lambda > 0$,
- (iii) the $n \times n$ Hessian matrix $[g_{ij}(x, y)] = [\frac{1}{2} \frac{\partial^2 F^2}{\partial y^i \partial y^j}]$ is positive definite at every point $(x, y) \in TM^0$.

In 1941, Randers introduced an important type of Finsler metrics by using Riemannian metrics and 1-forms on manifolds [23]. Randers metrics are as follows:

$$F(x, y) = \sqrt{g_{ij}(x)y^i y^j} + b_i(x)y^i, \tag{2.6}$$

where $g = (g_{ij}(x))$ is a Riemannian metric and $b = (b_i(x))$ is a nowhere zero 1-form on M . It has been shown that F is a Finsler metric if and only if $\|b\| = b_i(x)b^i(x) < 1$, where $b^i(x) = g^{ij}(x)b_j(x)$ and $[g^{ij}(x)]$ is the inverse matrix of $[g_{ij}(x)]$.

There is another way of describing Randers metrics. We can replace the 1-form $b = (b_i(x))$ with its dual, therefore Randers metrics can be defined as follows:

$$F(x, y) = \sqrt{g(x)(y, y)} + g(x)(X(x), y), \tag{2.7}$$

where X is a vector field on M such that $\|X\| = \sqrt{g(X, X)} < 1$.

F is of Berwald type if and only if X is parallel with respect to the Levi-Civita connection induced by the Riemannian metric g (see [3]).

A Riemannian metric g on the Lie group G is called left invariant if

$$g(x)(y, z) = g(e)(T_x l_{x^{-1}} y, T_x l_{x^{-1}} z) \quad \forall x \in G, \forall y, z \in T_x G, \tag{2.8}$$

where e is the unit element of G .

Suppose that g is a left invariant Riemannian metric on a Lie group G with Lie algebra \mathfrak{g} , then the Levi-Civita connection of g is defined by the following relation:

$$2\langle \nabla_U V, W \rangle = \langle [U, V], W \rangle - \langle [V, W], U \rangle + \langle [W, U], V \rangle, \tag{2.9}$$

for any $U, V, W \in \mathfrak{g}$, where $\langle \cdot, \cdot \rangle$ is the inner product induced by g on \mathfrak{g} .

We can define left invariant Finsler metrics similar to the Riemannian case. A Finsler metric is called left invariant if

$$F(x, y) = F(e, T_x l_{x^{-1}} y). \tag{2.10}$$

The simplest way for constructing left invariant Randers metrics on Lie groups is the use of left invariant Riemannian metrics and left invariant vector fields. Suppose that G is a Lie group, g is a left invariant Riemannian metric and X is a left invariant vector field such that $\sqrt{g(X, X)} < 1$, then we can define a left invariant Randers metric F as the formula (2.7).

An important quantity which associates with a Finsler space is flag curvature. This quantity is a natural generalization of the concept of sectional curvature in the Riemannian geometry which is computed by the following formula:

$$K(P, Y) = \frac{g_Y(R(U, Y)Y, U)}{g_Y(Y, Y) \cdot g_Y(U, U) - g_Y^2(Y, U)}, \tag{2.11}$$

where $g_Y(U, V) = \frac{1}{2} \frac{\partial^2}{\partial s \partial t} (F^2(Y + sU + tV))|_{s=t=0}$, $P = \text{span}\{U, Y\}$, $R(U, Y)Y = \nabla_U \nabla_Y Y - \nabla_Y \nabla_U Y - \nabla_{[U, Y]} Y$ and ∇ is the Chern connection induced by F (see [3, 28]).

From now on we suppose that G is a simply connected four-dimensional real Lie group.

3. Randers metrics of Berwald type on four-dimensional hypercomplex Lie groups

In this paper we consider left invariant hyper-Hermitian Riemannian metrics on left invariant hypercomplex four-dimensional simply connected Lie groups. These spaces have been classified by Barberis as follows (for more details see [4]).

Let G be a Lie group as above with Lie algebra \mathfrak{g} . She has shown that \mathfrak{g} is either Abelian or isomorphic to one of the following Lie algebras:

- (1) $[Y, Z] = W, [Z, W] = Y, [W, Y] = Z, X$ central,
- (2) $[X, Z] = X, [Y, Z] = Y, [X, W] = Y, [Y, W] = -X,$
- (3) $[X, Y] = Y, [X, Z] = Z, [X, W] = W,$
- (4) $[X, Y] = Y, [X, Z] = \frac{1}{2}Z, [X, W] = \frac{1}{2}W, [Z, W] = \frac{1}{2}Y,$

where $\{X, Y, Z, W\}$ is an orthonormal basis and in any case when we do not write a commutator between the elements of the basis, the commutator is zero.

Case (1) is diffeomorphic to $\mathbb{R} \times S^3$ and the other cases are diffeomorphic to \mathbb{R}^4 (see [4, 5]).

Now we discuss left invariant Randers metrics of Berwald type which can arise from these Riemannian metrics and left invariant vector fields on these spaces.

We begin with the Abelian case. For this case we have the following theorem (see [26]):

Theorem 3.1. *Let G be an Abelian Lie group equipped with a left invariant Riemannian metric g and let \mathfrak{g} be the Lie algebra of G . Suppose that $X \in \mathfrak{g}$ is a left invariant vector field with $\sqrt{g(X, X)} < 1$. Then the Randers metric F defined by the formula (2.7) is a flat geodesically complete locally Minkowskian metric on G .*

Now we continue with the other four cases.

Case 1. We can compute the Levi-Civita connection by using formula (2.9). (Also you can see [27].) The Levi-Civita connection is as follows:

$$\begin{aligned}
 \nabla_X X &= 0, & \nabla_X Y &= 0, & \nabla_X Z &= 0, & \nabla_X W &= 0, \\
 \nabla_Y X &= 0, & \nabla_Y Y &= 0, & \nabla_Y Z &= \frac{1}{2}W, & \nabla_Y W &= -\frac{1}{2}Z, \\
 \nabla_Z X &= 0, & \nabla_Z Y &= -\frac{1}{2}W, & \nabla_Z Z &= 0, & \nabla_Z W &= \frac{1}{2}Y, \\
 \nabla_W X &= 0, & \nabla_W Y &= \frac{1}{2}Z, & \nabla_W Z &= -\frac{1}{2}Y, & \nabla_W W &= 0.
 \end{aligned}
 \tag{3.1}$$

A simple computation shows that the only family of vector fields which is parallel with respect to this connection is of the form $Q = qX$ for any $q \in \mathbb{R}$. Now let $0 < \|Q\| < 1$ or equivalently let $0 < |q| < 1$, therefore by using these left invariant vector fields Q and formula (2.7), G admits a family of Randers metrics of Berwald type.

Case 2. The formula (2.9) shows that the Levi-Civita connection of the Riemannian metric of G is of the form [27]:

$$\begin{aligned}
 \nabla_X X &= -Z, & \nabla_X Y &= 0, & \nabla_X Z &= X, & \nabla_X W &= 0, \\
 \nabla_Y X &= 0, & \nabla_Y Y &= -Z, & \nabla_Y Z &= Y, & \nabla_Y W &= 0, \\
 \nabla_Z X &= 0, & \nabla_Z Y &= 0, & \nabla_Z Z &= 0, & \nabla_Z W &= 0, \\
 \nabla_W X &= -Y, & \nabla_W Y &= X, & \nabla_W Z &= 0, & \nabla_W W &= 0.
 \end{aligned}
 \tag{3.2}$$

Similar to case 1 by a simple computation we can show that the only parallel vector fields with respect to ∇ are of the form $Q = qW, q \in \mathbb{R}$. For constructing a non-Riemannian family of

Randers metrics of Berwald type defined by formula (2.7), it is sufficient to let $0 < \|Q\| < 1$ or equivalently $0 < |q| < 1$.

Cases 3 and 4. The Levi-Civita connections of cases 3 and 4 are of the following forms, respectively:

$$\begin{aligned} \nabla_X X &= 0, & \nabla_X Y &= 0, & \nabla_X Z &= 0, & \nabla_X W &= 0, \\ \nabla_Y X &= -Y, & \nabla_Y Y &= X, & \nabla_Y Z &= 0, & \nabla_Y W &= 0, \\ \nabla_Z X &= -Z, & \nabla_Z Y &= 0, & \nabla_Z Z &= X, & \nabla_Z W &= 0, \\ \nabla_W X &= -W, & \nabla_W Y &= 0, & \nabla_W Z &= 0, & \nabla_W W &= X, \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} \nabla_X X &= 0, & \nabla_X Y &= 0, & \nabla_X Z &= 0, & \nabla_X W &= 0, \\ \nabla_Y X &= -Y, & \nabla_Y Y &= X, & \nabla_Y Z &= -\frac{1}{4}W, & \nabla_Y W &= \frac{1}{4}Z, \\ \nabla_Z X &= -\frac{1}{2}Z, & \nabla_Z Y &= -\frac{1}{4}W, & \nabla_Z Z &= \frac{1}{2}X, & \nabla_Z W &= \frac{1}{4}Y, \\ \nabla_W X &= -\frac{1}{2}W, & \nabla_W Y &= \frac{1}{4}Z, & \nabla_W Z &= -\frac{1}{4}Y, & \nabla_W W &= \frac{1}{2}X. \end{aligned} \tag{3.4}$$

It is easy to show that these connections do not admit any parallel left invariant vector field; therefore there is not any left invariant Randers metric of Berwald type arising from left invariant vector fields (by using formula (2.7)) on these Riemannian Lie groups.

4. Flag curvature

In this section we discuss the flag curvature of invariant Randers metrics of cases 1 and 2.

Case 1. By using Levi-Civita connection for a curvature tensor we have [27]:

$$\begin{aligned} R(Y, Z)Y &= -R(Z, W)W = -\frac{1}{4}Z, \\ R(Y, W)W &= R(Y, Z)Z = \frac{1}{4}Y, \\ R(Z, W)Z &= R(Y, W)Y = -\frac{1}{4}W, \end{aligned} \tag{4.1}$$

and in other cases $R = 0$. Now let $U = aX + bY + cZ + dW$ and $V = \tilde{a}X + \tilde{b}Y + \tilde{c}Z + \tilde{d}W$ be two arbitrary vectors in \mathfrak{g} then we have

$$\begin{aligned} R(V, U)U &= -\frac{1}{4}\{(b\tilde{c} - c\tilde{b})(cY - bZ) + (b\tilde{d} - d\tilde{b})(dY - bW) \\ &\quad + (c\tilde{d} - d\tilde{c})(dZ - cW)\}. \end{aligned} \tag{4.2}$$

Since F is of Berwald type therefore the curvature tensors of F and g coincide. Suppose that $\{U, V\}$ is an orthonormal basis for $P = \text{span}\{U, V\}$ with respect to the inner product \langle, \rangle induced by g . Now by using the formula $g_U(V_1, V_2) = \frac{1}{2} \frac{\partial^2}{\partial s \partial t} (F^2(U + sV_1 + tV_2))|_{s=t=0}$ of F (for an explicit formula you can see [14]) we have

$$g_U(R(V, U)U, V) = \frac{1}{4}(1 + aq)\{(b\tilde{c} - c\tilde{b})^2 + (b\tilde{d} - d\tilde{b})^2 + (c\tilde{d} - d\tilde{c})^2\} \tag{4.3}$$

$$g_U(U, U) = (1 + aq)^2 \tag{4.4}$$

$$g_U(V, V) = 1 + aq + (\tilde{a}q)^2 \tag{4.5}$$

$$g_U(U, V) = \tilde{a}q(1 + aq). \tag{4.6}$$

Now by using (2.11) we have

$$K(P, U) = \frac{(b\tilde{c} - c\tilde{b})^2 + (b\tilde{d} - d\tilde{b})^2 + (c\tilde{d} - d\tilde{c})^2}{4(1 + aq)^2} \geq 0. \tag{4.7}$$

Therefore in case 1 (G, F) is of a non-negative flag curvature.

Case 2. The curvature tensor of Riemannian metric (Finsler metric) of this case is of the form

$$\begin{aligned} R(X, Y)X &= -R(Y, Z)Z = Y, \\ R(X, Y)Y &= R(X, Z)Z = -X, \\ R(X, Z)X &= R(Y, Z)Y = Z. \end{aligned} \tag{4.8}$$

In this case for any U and V we have

$$R(V, U)U = -\{(a\tilde{b} - b\tilde{a})(aY - bX) + (a\tilde{c} - c\tilde{a})(aZ - cX) + (b\tilde{c} - c\tilde{b})(bZ - cY)\}, \tag{4.9}$$

Let $P = \{U, V\}$ be as case 1. Therefore for the Randers metric F described in case 2 we have

$$g_U(R(V, U)U, V) = -(1 + dq)\{(a\tilde{b} - b\tilde{a})^2 + (a\tilde{c} - c\tilde{a})^2 + (b\tilde{c} - c\tilde{b})^2\} \tag{4.10}$$

$$g_U(U, U) = (1 + dq)^2 \tag{4.11}$$

$$g_U(V, V) = 1 + dq + (\tilde{d}q)^2 \tag{4.12}$$

$$g_U(U, V) = \tilde{d}q(1 + dq). \tag{4.13}$$

Hence for the flag curvature we have

$$K(P, U) = \frac{-\{(a\tilde{b} - b\tilde{a})^2 + (a\tilde{c} - c\tilde{a})^2 + (b\tilde{c} - c\tilde{b})^2\}}{(1 + dq)^2} \leq 0, \tag{4.14}$$

which shows that (G, F) is of a non-positive flag curvature.

Remark 4.1. The metrics constructed in cases 1 and 2 are complete and J_i -invariant, $i = 1, 2, 3$.

Proof. These metrics clearly are J_i -invariant, $i = 1, 2, 3$, so we prove the completeness. Since the metric F (in case 1 or 2) is of Berwald type therefore the geodesics of F and g coincide. On the other hand (G, g) is a homogeneous Riemannian manifold, hence (G, g) is geodesically complete (see [9], p 185). Therefore (G, F) is geodesically complete. Now the Hopf–Rinow theorem and connectedness of G will complete the proof. \square

Acknowledgments

This work was supported by a research grant from the Shahrood University of Technology.

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